DIGITAL SIGNAL PROCESSING

UNIT-II
Z Transform
Realization of Discrete Systems

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Introduction to Z Transform (ZT):

Z Transform is a mathematical tool, which is used to evaluate Z domain representation of a discrete time domain sequence.

Z Transform of a discrete time signal or sequence x(n) is represented with X(z) and it can be evaluated by using the mathematical formula

$$ZT[\ x(n)\ ] = X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \text{ --------(1)}$$

Above equation (1) is called bi-directional or both sided Z Transform, because x(n) is both-sided.

If x(n) is causal or right sided, then its Z transform can be defined as

$$ZT[\ x(n)\ ] = X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} \text{ --------(2)}$$

If x(n) is anti-causal or left sided, then its Z transform can be defined as

$$ZT[\ x(n)\ ] = X(z) = \sum_{n=-\infty}^{-1} x(n)z^{-n} \text{ --------(3)}$$

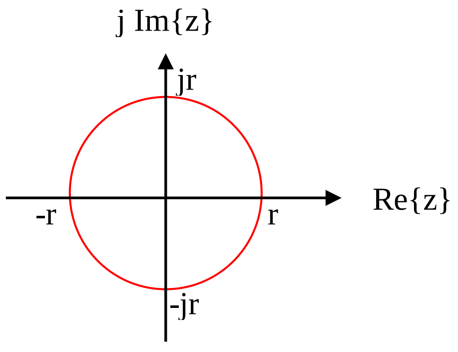
Above equations (2) and (3) are called uni-directional or one-sided Z transform.

Where, z is a complex variable and it can be defined as

$$z = r\, e^{j\omega}$$
$$= r\cos(\omega) + j\, r\sin(\omega)$$
$$= Re\{ z \} + j\, Im\{ z \}$$

Where, r is magnitude of z and ω is phase of z or digital frequency, measured in rad/sample.

A graph, which is drawn between Re{ z } = r cos(ω) on x-axis and jIm{ z } = jr sin(ω) on y-axis is called z-plane.



ω	Re{z}=rcos(ω)	Im{z}=jrsin(ω)	$\lvert z \rvert$
0^0	r	j0	r
90^0	0	jr	r
180^0	– r	j0	r
270^0	0	–jr	r

z-plane is a circle, centered about origin with a radius of $\lvert z \rvert$

Relation between ZT and DTFT:

From the basic definition of Z Transform

$$ZT[\ x(n)\ ] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\left(re^{j\omega}\right)^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\ r^{-n} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} [\ r^{-n}\ x(n)\ ]\ e^{-j\omega n}$$

$$ZT[\ x(n) = DTFT\ [\ r^{-n}\ x(n)\ ]$$

If r = 1, then ZT [x(n)] = DTFT [x(n)].

On the unit circle of z-plane, both the ZT and DTFT are same.

Z Transform of various classes of Signals:

Various classes of signals or sequences are given below

- Right-sided signal with infinite duration
- Left-sided signal with infinite duration
- Both-sided signal with infinite duration
- Finite duration signal

(A)Z Transform of a Right-sided Signal with Infinite Duration:

In general, right-sided with infinite duration signals are extending from 0 to ∞.

Example: $x(n) = a^{n} u(n)$

From the basic definition of z transform

$$ZT[\ x(n)\ ] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$ZT[\ a^{n}\ u(n)\ ] = \sum_{n=-\infty}^{\infty} a^{n}\ u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^{n}\ z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^{n}$$

$$= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^{2} + \left(\frac{a}{z}\right)^{3} + \ldots\ldots. + \left(\frac{a}{z}\right)^{\infty}$$

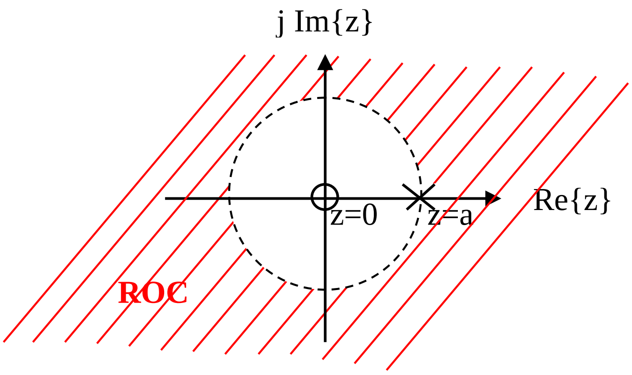
$$= \frac{1}{1-\frac{a}{z}}\ ,\ \text{if}\ \left|\frac{a}{z}\right| < 1$$

$$= \frac{1}{\frac{z-a}{z}}, \quad |a|<|z|$$

$$X(z) = \frac{z}{z-a}, \quad |z|>|a|$$

	ROC
$ZT[a^n u(n)] = X(z) = \dfrac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$

- X(z) has one zero, which is located at z=0 and one pole, which is located at z=a.
- If we locate poles and zeros on the z-plane, then the z-plane is called pole-zero plot.
- The range of values of z for which X(z) is finite is called Region of Convergence (ROC).
- Now draw a z-plane and indicate ROC, poles and zeros.



If the signal x(n) is right-sided with infinite duration, then its ROC is outside the circle of outer most pole.

(B)Z Transform of a Anti-causal or left-sided Signal with Infinite Duration:

In general, left-sided with infinite duration signals are extending from -∞ to -1.

Example: $x(n) = -a^n u(-n-1)$

From the basic definition of z transform

$$ZT[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$ZT[-a^n u(-n-1)] = \sum_{n=-\infty}^{\infty} [-a^n u(-n-1)] z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} \left(\frac{a}{z}\right)^n$$

$$= -\sum_{n=1}^{\infty} \left(\frac{z}{a}\right)^n$$

$$= -\left[\left(\frac{z}{a}\right) + \left(\frac{z}{a}\right)^2 + \left(\frac{z}{a}\right)^3 + ... + \left(\frac{z}{a}\right)^\infty\right]$$

$$= -\left(\frac{z}{a}\right)\left[1 + \left(\frac{z}{a}\right) + \left(\frac{z}{a}\right)^2 + \left(\frac{z}{a}\right)^3 + ... + \left(\frac{z}{a}\right)^\infty\right]$$

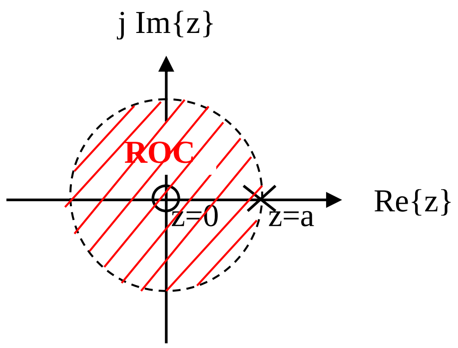
$$= -\left(\frac{z}{a}\right)\left(\frac{1}{1-\frac{z}{a}}\right), \text{ if } \left|\frac{z}{a}\right| < 1$$

$$= -\left(\frac{z}{a}\right)\left(\frac{a}{a-z}\right), \text{ if } |z| < |a|$$

$$= \frac{z}{z-a}, \quad |z| < |a|$$

	ROC
$ZT[-a^n u(-n-1)] = X(z) = \frac{z}{z-a}$	$\lvert z \rvert < \lvert a \rvert$

- X(z) has one zero, which is located at z=0 and one pole, which is located at z=a.
- If we locate poles and zeros on the z-plane, then the z-plane is called pole-zero plot.
- The range of values of z for which X(z) is finite is called Region of Convergence (ROC).
- Now draw a z-plane and indicate ROC, poles and zeros.



If the signal x(n) is left-sided with infinite duration, then its ROC is inside the circle of innermost pole.

(C)Z Transform of Both-sided Signal with Infinite Duration:

In general, both-sided signals with infinite duration are extending from -∞ to ∞.

Example: $x(n) = a^{|n|}$

From the basic definition of z transform

$$ZT[\ x(n)\ ] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$ZT[\ a^{|n|}\ ] = \sum_{n=-\infty}^{\infty} a^{|n|}\ z^{-n}$$

$$= \sum_{n=-\infty}^{-1} a^{|n|}\ z^{-n} + \sum_{n=0}^{\infty} a^{|n|}\ z^{-n}$$

$$= \sum_{n=-\infty}^{-1} a^{-n}\ z^{-n} + \sum_{n=0}^{\infty} a^{n}\ z^{-n}$$

$$= \sum_{n=-\infty}^{-1} (az)^{-n} + \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^{n}$$

$$= \sum_{n=1}^{\infty} (az)^{n} + \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^{n}$$

$$= \left[az + (az)^2 + (az)^3 + ... + (az)^{\infty}\right] + \left[1 + \left(\frac{a}{z}\right)^1 + \left(\frac{a}{z}\right)^2 + ... + \left(\frac{a}{z}\right)^{\infty}\right]$$

$$= az\left[1 + az + (az)^2 + (az)^3 + ... + (az)^{\infty}\right] + \left[1 + \left(\frac{a}{z}\right)^1 + \left(\frac{a}{z}\right)^2 + ... + \left(\frac{a}{z}\right)^{\infty}\right]$$

$$= az\left[\frac{1}{1-az}\right] + \left[\frac{1}{1-\frac{a}{z}}\right], \quad \text{if } |az| < 1 \ \ \& \ \left|\frac{a}{z}\right| < 1$$

$$= \left[\frac{az}{1-az}\right] + \left[\frac{z}{z-a}\right], \quad \text{if } |z| < 1/|a| \ \ \& \ |a| < |z|$$

$$= \left[\frac{az(z-a) + z(1-az)}{(1-az)(z-a)}\right], \quad \text{if } |z| < 1/|a| \ \ \& \ |z| > |a|$$

$$= \left[\frac{z(az - a^2 + 1 - az)}{(1-az)(z-a)}\right], \quad |a| < |z| < 1/|a|$$

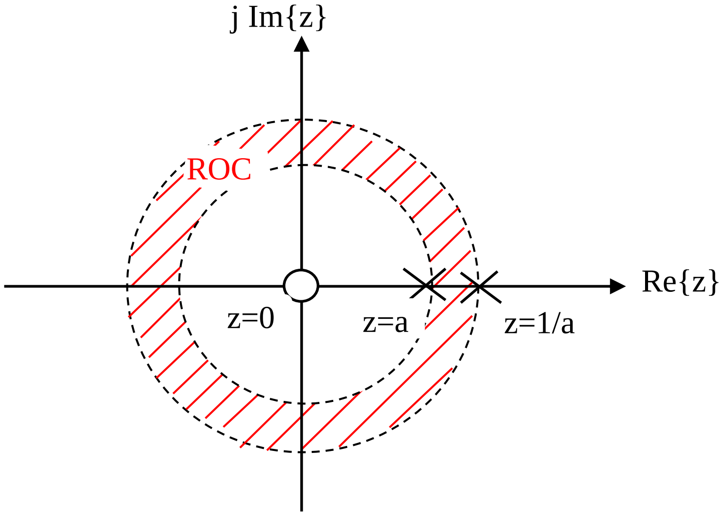
$$= \left[\frac{z(1-a^2)}{(1-az)(z-a)}\right], \quad |a| < |z| < 1/|a|$$

$$= \left[\frac{z(1-a^2)}{-a(z-1/a)(z-a)}\right], \quad |a| < |z| < 1/|a|$$

$$= \left[\frac{z(a-1/a)}{(z-1/a)(z-a)}\right], \quad |a| < |z| < 1/|a|$$

$ZT[a^{\lfloor n \rfloor}] = X(z) = \dfrac{z\left(a-\frac{1}{a}\right)}{(z-a)\left(z-\frac{1}{a}\right)}$	ROC
	$\lvert a \rvert < \lvert z \rvert < \dfrac{1}{\lvert a \rvert}$

- X(z) has one zero, which is located at z=0 and two poles, which is located at z=a and z=1/a.
- If we locate poles and zeros on the z-plane, then the z-plane is called pole-zero plot.
- The range of values of z for which X(z) is finite is called Region of Convergence (ROC).
- Now draw a z-plane and indicate ROC, poles and zeros.



If the signal x(n) is both-sided with infinite duration, then its ROC is a finite duration ring, which lies between two poles.

(D)Z Transform of Finite duration Signal:

In general, a finite duration signal consists of finite number of samples.

Example-1: x(n) = {1}

From the basic definition of z transform

$$ZT[\, x(n)\,] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$
$$= x(0) z^{-0}$$
$$= 1 \times 1$$
$$= 1$$

$ZT[x(n)] = X(z) = 1$	ROC
	Entire z-plane

Example-2: x(n) = {1,-1}

From the basic definition of z transform

$$ZT[\, x(n)\,] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$
$$= x(0) z^{-0} + x(1) z^{-1}$$
$$= 1 \times 1 - 1 \times z^{-1}$$
$$= 1 - z^{-1}$$
$$= \frac{z-1}{z}$$

$ZT[x(n)] = X(z) = \frac{z-1}{z}$	ROC
	Entire z-plane except z=0

Example-3: $x(n) = \{1, -\underset{\uparrow}{1}\}$

From the basic definition of z transform

$$ZT[\, x(n)\,] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$
$$= x(-1) z^{-(-1)} + x(0) z^{-0}$$
$$= 1 \times z - 1 \times z^{0}$$
$$= z - 1$$

$ZT[x(n)] = X(z) = z - 1$	ROC
	Entire z-plane except z=±∞

Region of Convergence (ROC) and its Properties:

The range of values of z for which the basic definition of z transform will converges or produces a finite result is called Region of Convergence (ROC).

Property-1:

If x(n) is right-sided sequence with infinite duration, then its ROC is outside the circle of outermost pole.

Example: $x(n) = 3\left(\frac{1}{2}\right)^n u(n) - 2\left(\frac{1}{3}\right)^n u(n)$

$$ZT[\,x(n)\,] = ZT\left[3\left(\frac{1}{2}\right)^n u(n) - 2\left(\frac{1}{3}\right)^n u(n)\right]$$

$$X(z) = 3ZT\left[\left(\frac{1}{2}\right)^n u(n)\right] - 2ZT\left[\left(\frac{1}{3}\right)^n u(n)\right]$$

$$= \frac{3z}{z-\frac{1}{2}} - \frac{2z}{z-\frac{1}{3}} \quad , \; |z| > \frac{1}{2} \;\&\; |z| > \frac{1}{3}$$

$$= \frac{z(3z-1-2z+1)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}, \; |z| > \frac{1}{2}$$

$$= \frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}, \; |z| > \frac{1}{2}$$

	ROC
$ZT\left[3\left(\frac{1}{2}\right)^n u(n) - 2\left(\frac{1}{3}\right)^n u(n)\right] = X(z) = \frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$	$\|z\| > \frac{1}{2}$

Property-2:

If x(n) is left-sided sequence with infinite duration, then its ROC is inside the circle of innermost pole.

Example: $x(n) = 3(2)^n u(-n-1) - 2(3)^n u(-n-1)$

$$ZT[\,x(n)\,] = ZT\left[3(2)^n u(-n-1) - 2(3)^n u(-n-1)\right]$$

$$X(z) = 3ZT\left[(2)^n u(-n-1)\right] - 2ZT\left[(3)^n u(-n-1)\right]$$

$$= 3ZT\left[(2)^n u(-n-1)\right] - 2ZT\left[(3)^n u(-n-1)\right]$$

$$= \frac{-3z}{z-2} - \frac{-2z}{z-3} \quad , \; |z| < 2 \;\&\; |z| < 3$$

$$= \frac{-z(3z-9-2z+4)}{(z-2)(z-3)}, \; |z| < 2$$

$$X(z) = \frac{-z(z-5)}{(z-2)(z-3)}, \; |z| < 2$$

$ZT\left[3(2)^n u(-n-1) - 2(3)^n u(-n-1)\right] = X(z) = \frac{-z(z-5)}{(z-2)(z-3)}$	ROC
	$\|z\| < 2$

Property-3:

If x(n) is both-sided sequence with infinite duration, then its ROC is a finite duration ring, which lies between two poles.

Example: $x(n) = 3\left(\frac{1}{2}\right)^n u(n) - 2(3)^n u(-n-1)$

$$ZT[\,x(n)\,] = ZT\left[3\left(\frac{1}{2}\right)^n u(n) - 2(3)^n u(-n-1)\right]$$

$$X(z) = 3ZT\left[\left(\frac{1}{2}\right)^n u(n)\right] - 2ZT\left[(3)^n u(-n-1)\right]$$

$$= \frac{3z}{z-\frac{1}{2}} - \frac{2(-z)}{z-3} \;, \; |z| > \frac{1}{2} \;\&\; |z| < 3$$

$$= \frac{z(3z-9+2z-1)}{\left(z-\frac{1}{2}\right)(z-3)}, \; \frac{1}{2} < |z| < 3$$

$$= \frac{5z(z-2)}{\left(z-\frac{1}{2}\right)(z-3)}, \; \frac{1}{2} < |z| < 3$$

$ZT\left[3\left(\frac{1}{2}\right)^n u(n) - 2(3)^n u(-n-1)\right] = X(z) = \frac{5z(z-2)}{\left(z-\frac{1}{2}\right)(z-3)}$	ROC
	$\frac{1}{2} < \|z\| < 3$

Property-4:

If x(n) is finite duration sequence, then its ROC is entire z-plane except possibly z=0 and/or z=±∞.

Example-1: x(n) = {1}

$$ZT[\,x(n)\,] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= x(0) z^{-0}$$

$$= 1 \times 1$$

$$= 1$$

$ZT[x(n)] = X(z) = 1$	ROC
	Entire z-plane

Example-2: x(n) = {1,-1}

$$ZT[\ x(n)\ ] = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$
$$= x(0)z^{-0} + x(1)z^{-1}$$
$$= 1\times1 - 1\times z^{-1}$$
$$= 1 - z^{-1}$$
$$= \frac{z-1}{z}$$

$ZT[x(n)] = X(z) = \frac{z-1}{z}$	ROC
	Entire z-plane except z=0

Example-3: $x(n) = \{1, -\underset{\uparrow}{1}\}$

$$ZT[\ x(n)\ ] = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$
$$= x(-1)z^{-(-1)} + x(0)z^{-0}$$
$$= 1\times z - 1\times z^{0}$$
$$= z - 1$$

$ZT[x(n)] = X(z) = z-1$	ROC
	Entire z-plane except z=±∞

Property-5:

Within the ROC, poles do not exist.

Example: Above all examples.

Property-6:

ROC is independent of zero's.

Example: Above all examples.

Properties of Z Transform:

(A)Linear Property:

If $x_1(n)$, $x_2(n)$ are two discrete time sequences and ZT[$x_1(n)$] = $X_1(z)$, ZT[$x_2(n)$] = $X_2(z)$, then ZT[a $x_1(n)$ + b $x_2(n)$] = a $X_1(z)$ + b $X_2(z)$ is called linear property of z transform

Proof:

From the basic definition of z transform of a sequence x(n)

$$ZT[\ x(n)\ ] = \sum_{n=-\infty}^{\infty} x(n) z^{-n},$$

replace x(n) with a $x_1(n)$ + b $x_2(n)$

$$\begin{aligned}
ZT[\ a\,x_1(n) + b\,x_2(n)] &= \sum_{n=-\infty}^{\infty} [\,a\,x_1(n) + b\,x_2(n)\,]\,z^{-n} \\
&= \sum_{n=-\infty}^{\infty} [\,a\,x_1(n) + b\,x_2(n)\,]\,z^{-n} \\
&= \sum_{n=-\infty}^{\infty} [\,a\,x_1(n)\,z^{-n} + b\,x_2(n)\ z^{-n}] \\
&= \sum_{n=-\infty}^{\infty} [\,a\,x_1(n)\,z^{-n}] + \sum_{n=-\infty}^{\infty} [\,b\,x_2(n)\ z^{-n}] \\
&= a\sum_{n=-\infty}^{\infty} x_1(n)\,z^{-n} + b\sum_{n=-\infty}^{\infty} [\,x_2(n)\ z^{-n}] \\
&= a\,ZT[\ x_1(n)\,] + b\,ZT[\ x_2(n)\,] \\
&= a\,X_1(z) + b\,X_2(z)
\end{aligned}$$

Example: $x_1(n) = \left(\frac{1}{2}\right)^n u(n)$ and $x_2(n) = \left(\frac{1}{3}\right)^n u(n)$

$$\begin{aligned}
ZT[\ 3x_1(n) + 2x_2(n)\ ] &= 3ZT[\ x_1(n)] + 2ZT[x_2(n)\ ] \\
&= 3ZT\left[\left(\frac{1}{2}\right)^n u(n)\right] + 2ZT\left[\left(\frac{1}{3}\right)^n u(n)\right] \\
&= \frac{3z}{z-\frac{1}{2}} + \frac{2z}{z-\frac{1}{3}}\ ,\ |\,z\,| > \frac{1}{2} \,\&\, |\,z\,| > \frac{1}{3} \\
&= \frac{z(3z-1+2z-1)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)},\ |\,z\,| > \frac{1}{2} \\
&= \frac{z(5z-2)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)},\ |\,z\,| > \frac{1}{2}
\end{aligned}$$

(B)Time Shifting Property:

If x(n) is a discrete time sequence and ZT[x(n)] = X(z),

then ZT[$x(n - n_0)$] = z^{-n_0} X(z) is called time shifting property of z transform.

Proof:

From the basic definition of z transform of a sequence x(n)

$$ZT[\ x(n)\ ] = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \text{ , replace x(n) with } x(n-n_0)$$

$$ZT[\ x(n)\ ] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$ZT[\ x(n-n_0)\ ] = \sum_{n=-\infty}^{\infty} x(n-n_0) z^{-n}, \text{Let } n - n_0 = m \Rightarrow n = n_0 + m$$

$$= \sum_{m=-\infty}^{\infty} x(m) z^{-(n_0+m)}$$

$$= \sum_{m=-\infty}^{\infty} x(m) z^{-n_0}\ z^{-m}$$

$$= z^{-n_0} \sum_{m=-\infty}^{\infty} x(m)\ z^{-m}$$

$$= z^{-n_0} ZT\ [\ x(n)\ ]$$

$$= z^{-n_0} X(z)$$

Example:

$$ZT\left[3\left(\frac{1}{2}\right)^{n+1} u(n-9)\right] = 3ZT\left[\left(\frac{1}{2}\right)^{n-9+10} u(n-9)\right]$$

$$= 3\left(\frac{1}{2}\right)^{10} ZT\left[\left(\frac{1}{2}\right)^{n-9} u(n-9)\right]$$

$$= 3\left(\frac{1}{2}\right)^{10} z^{-9} ZT\left[\left(\frac{1}{2}\right)^{n} u(n)\right]$$

$$= 3\left(\frac{1}{2}\right)^{10} z^{-9}\left(\frac{z}{z-\frac{1}{2}}\right), \ |z| > \frac{1}{2}$$

$$= \frac{3\left(\frac{1}{2}\right)^{10}}{z^{8}\left(z-\frac{1}{2}\right)}, \ |z| > \frac{1}{2}$$

Note: If initial conditions are considered, then

- $ZT[\ x(n-1)\ ] = z^{-1}X(z) + x(-1)$
- $ZT[\ x(n-2)\ ] = z^{-2}X(z) + z^{-1}x(-1) + x(-2)$

(C)Time Reversal Property:

If x(n) is a discrete time sequence and ZT[x(n)] = X(z),

then ZT[x(– n)] = X(1/z) is called time reversal property of z transform.

Proof:

From the basic definition of z transform of a sequence x(n)

$$ZT[\ x(n)\ ] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Replace x(n) with x(– n)

$$ZT[\ x(-n)\ ] = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}, \text{Let } n = -m, n = -m,$$

$$= \sum_{m=\infty}^{-\infty} x(m) z^{-(-m)}$$

$$= \sum_{m=\infty}^{-\infty} x(m)\ \ (z^{-1})^{-m}$$

$$= \sum_{m=\infty}^{-\infty} x(m)\ \left(\frac{1}{z}\right)^{-m}$$

$$= ZT[\ x(n)\ ]\ \text{ with replacement of } z = 1/z$$

$$= X\left(\frac{1}{z}\right)$$

Example:

Evaluate the z transform of u(n) and u(-n)

We know that,

$$ZT[a^n u(n)] = \frac{z}{z-a} \quad , \ |z| > a$$

Put a = 1

$$ZT[u(n)] = \frac{z}{z-1} \quad , \ |z| > 1$$

Apply time shifting property of z Transform

$$ZT[u(-n)] = \frac{1/z}{1/z-1} \quad , \ |1/z| > 1$$

$$= \frac{1}{1-z} \quad , \ |z| < 1$$

(D)Conjugate Property:

If x(n) is discrete time sequence and ZT[x(n)] = X(z),

then ZT[x*(n)] = X*(z*) is conjugate property of z transform.

Proof:

From the basic definition of z transform of a sequence x(n)

$$ZT[\ x(n)\ ] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Replace x(n) with x*(n)

$$ZT[\ x^*(n)\ ] = \sum_{n=-\infty}^{\infty} x^*(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x^*(n) \left((z^*)^{-n}\right)^*$$

$$= \sum_{n=-\infty}^{\infty} [\ x(n)\ (z^*)^{-n}\ ]^*$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n)\ (z^*)^{-n}\right]^*$$

$$= (ZT\ [\ x(n)\ ] \text{ with } z = z^*)^*$$

$$= (X(z) \text{ with } z = z^*)^*$$

$$= [X(z^*)]^*$$

$$= X^*(z^*)$$

(E)Exponential or Scaling in z-domain Property:

If x(n) is a discrete time sequence and ZT[x(n)] = X(z),

then ZT[a^n x(n)] = X(z/a) is called exponential or scaling in z-domain property.

Proof:

From the basic definition of z transform of a sequence x(n)

$$ZT[\ x(n)\ ] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Replace x(n) with a^n x(n)

$$ZT[\ a^n\ x(n)\ ] = \sum_{n=-\infty}^{\infty} a^n\ x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\ a^n\ z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left(\frac{z}{a}\right)^{-n}$$

$$= X\left(\frac{z}{a}\right)$$

(F)Multiplication by n or Differentiation in z-domain Property:

If x(n) is a discrete time sequence and ZT[x(n)] = X(z),

then ZT[nx(n)]= - d/dz [X(z)] is called multiplication by n or differentiation in z domain property.

Proof:

From the basic definition of z transform of a sequence x(n)

$$ZT[\ x(n)\ ] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Differentiate w.r.t z

$$\frac{d}{dz}[\ X(z)\ ] = \sum_{n=-\infty}^{\infty} x(n)\ \frac{d}{dz} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\ (-n) z^{-n-1}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\ (-n) z^{-n} z^{-1}$$

$$= -z^{-1} \sum_{n=-\infty}^{\infty} [\ n\ x(n)\ ]\ z^{-n}$$

$$\frac{d}{dz}[\ X(z)\ ] = -\frac{1}{z} ZT[\ nx(n)\ ]$$

$$ZT[\ nx(n)\ ] = -z \frac{d}{dz}[\ X(z)\ ]$$

Example:

Evaluate the z transform of $na^n u(n)$

We know that,

$$ZT[a^n u(n)] = \frac{z}{z-a} \quad , \ |z| > a$$

Apply differentiation in z domain property

$$ZT[na^n u(n)] = -z \frac{d}{dz}\left(\frac{z}{z-a}\right)$$

$$= -z\left(\frac{(z-a)(1) - z(1)}{(z-a)^2}\right)$$

$$= -z\left(\frac{z-a-z}{(z-a)^2}\right)$$

$$= \frac{az}{(z-a)^2}$$

(G)Initial Value Theorem:

If x(n) is a discrete time causal sequence and ZT[x(n)] = X(z), then the initial value of a causal signal can be computed from x(n) as well as X(z) by using the formula

$$x(0) = \underset{n \to 0}{\text{Lt}}\, x(n) = \underset{z \to \infty}{\text{Lt}}\, X(z)$$ is called initial value theorem.

Proof:

From the basic definition of z transform of a sequence x(n)

$$\text{ZT}[\ x(n)\ ] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots\dots\dots$$

$$= x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \dots\dots\dots$$

Apply as limit $z \to \infty$

$$\underset{z \to \infty}{\text{Lt}}\, X(z) = x(0) + \frac{x(1)}{\infty} + \frac{x(2)}{\infty^2} + \dots\dots\dots$$

$$= x(0) + 0 + 0 + \dots\dots\dots\dots$$

$$= x(0)$$

$$\Rightarrow x(0) = \underset{n \to 0}{\text{Lt}}\, x(n) = \underset{z \to \infty}{\text{Lt}}\, X(z)$$

Example:

Evaluate the initial value of a causal signal x(n) from the z domain $X(z) = \dfrac{z(5z-2)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)},\ |z| > \dfrac{1}{2}$

$$x(0) = \underset{z \to \infty}{\text{Lt}}\, X(z)$$

$$= \underset{z \to \infty}{\text{Lt}}\, \frac{z(5z-2)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

$$= \underset{z \to \infty}{\text{Lt}}\, \frac{5-\frac{2}{z}}{\left(1-\frac{1}{2z}\right)\left(1-\frac{1}{3z}\right)}$$

$$= \frac{5-0}{(1-0)(1-0)}$$

$$= 5$$

(H)Final Value Theorem:

If x(n) is a discrete time causal sequence and ZT[x(n)] = X(z), then the final value of a causal signal can be computed from x(n) as well as X(z) by using the formula

$$x(\infty) = \underset{n \to \infty}{\text{Lt}}\, x(n) = \underset{z \to 1}{\text{Lt}}(1 - z^{-1})X(z) = \underset{z \to 1}{\text{Lt}}(z-1)X(z)$$ is called final value theorem.

Proof:

From the basic definition of z transform of a causal sequence x(n)

$$ZT[\ x(n)\ ] = \sum_{n=0}^{\infty} x(n) z^{-n}$$

Replace x(n) by x(n) – x(n – 1)

$$ZT[\ x(n) - x(n-1)] = \sum_{n=0}^{\infty} [\ x(n) - x(n-1)\ ] z^{-n}$$

$$X(z) - z^{-1}\ X(z) = \sum_{n=0}^{\infty} [\ x(n) - x(n-1)\ ] z^{-n}$$

$$(1 - z^{-1}\ )X(z) = \sum_{n=0}^{\infty} [\ x(n) - x(n-1)\ ] z^{-n}$$

Apply as $z \to 1$

$$\underset{z \to 1}{\text{Lt}}(1 - z^{-1}\ )X(z) = \underset{z \to 1}{\text{Lt}} \sum_{n=0}^{\infty} [\ x(n) - x(n-1)\ ] z^{-n}$$

$$= \sum_{n=0}^{\infty} [\ x(n) - x(n-1)\ ] \underset{z \to 1}{\text{Lt}}\, z^{-n}$$

$$= \sum_{n=0}^{\infty} [\ x(n) - x(n-1)\ ]$$

$$= [x(0) - x(-1)] + [x(1) - x(0)] + [x(2) - x(1)] + \dots\dots\dots.$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots + [x(\infty - 1) - x(\infty - 2)] + [x(\infty) - x(\infty - 1)]$$

$$= -x(-1) + x(\infty)$$

$$= x(\infty)$$

$$\Rightarrow x(\infty) = \underset{n \to \infty}{\text{Lt}}\, x(n) = \underset{z \to 1}{\text{Lt}}(1 - z^{-1}\ )X(z) = \underset{z \to 1}{\text{Lt}}(z-1)X(z)$$

Example:

Evaluate the final value of a causal signal x(n) from the z domain $X(z) = \dfrac{z(5z-2)}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)},\ |z| > \dfrac{1}{2}$

$$x(\infty) = \underset{z \to 1}{\text{Lt}}(z-1)X(z) == \underset{z \to 1}{\text{Lt}} \frac{(z-1)z(5z-2)}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)}$$

$$= \frac{(1-1)(1)(5-2)}{(1-1/2)(1-1/3)} = 0$$

(I)Time Convolution Theorem:

If $x_1(n)$, $x_2(n)$ are two discrete time sequences and ZT[$x_1(n)$] = $X_1(z)$, ZT[$x_2(n)$] = $X_2(z)$, then ZT[$x_1(n) * x_2(n)$] = $X_1(z)\, X_2(z)$ is called time convolution theorem.

Proof:

From the basic definition of z transform of a sequence x(n)

$$ZT[\ x(n)\ ] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Replace x(n) by $x_1(n) * x_2(n)$

$$ZT[\ x_1(n) * x_2(n)\ ] = \sum_{n=-\infty}^{\infty} [\ x_1(n) * x_2(n)\ ]\ z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} [\ x_1(m)\ x_2(n-m)\ ] \right) z^{-n}$$

change the order of summation

$$= \sum_{m=-\infty}^{\infty} x_1(m) \left(\sum_{n=-\infty}^{\infty} x_2(n-m)\ z^{-n} \right)$$

$$= \sum_{m=-\infty}^{\infty} x_1(m)\ \left(ZT[x_2(n-m)]\right)$$

$$= \sum_{m=-\infty}^{\infty} x_1(m)\ z^{-m} ZT[x_2(n)\ ]$$

$$= \sum_{m=-\infty}^{\infty} x_1(m)\ z^{-m} X_2(z)$$

$$= X_1(z) X_2(z)$$

Example:

Evaluate the z transform of $a^n u(n) * n a^n u(n)$

We know that,

$$ZT[a^n u(n)] = \frac{z}{z-a} \quad , \ |\ z\ | > a \text{ and}$$

$$ZT[na^n u(n)] = \frac{az}{(z-a)^2}, |\ z\ | > a$$

$$ZT[a^n u(n) * na^n u(n)] = ZT[a^n u(n)] ZT[na^n u(n)]$$

$$= \frac{z}{z-a} \frac{az}{(z-a)^2}$$

$$= \frac{az^2}{(z-a)^3}, |\ z\ | > a$$

Inverse Z Transform:

Inverse z transform is used to evaluate the discrete time sequence x(n) from the z-domain X(z) and its Region of Convergence (ROC). Various methods of Inverse z transform are given below.

- Partial Fractions Method
- Power Series Method or Long Division Method
- Residue Method or Contour Integral Method

(A)Partial Fractions Method:

If the z-domain X(z) is given, then take X(z)/z and split into partial fractions. Finally multiply with z and use the following formulas to obtain the time domain sequence x(n) in this partial fractions method.

- $ZT[\delta(n)] = 1 \Rightarrow Z^{-1}[1] = \delta(n)$
- $ZT[\delta(n-m)] = z^{-m} \Rightarrow Z^{-1}[z^{-m}] = \delta(n-m)$
- $ZT[x(n-m)] = z^{-m} X(z) \Rightarrow Z^{-1}[z^{-m} X(z)] = x(n-m)$
- $ZT[a^n u(n)] = \frac{z}{z-a}, |z| > a$; $ZT[-a^n u(-n-1)] = \frac{z}{z-a}, |z| < a$ $\Rightarrow Z^{-1}\left[\frac{z}{z-a}\right] = \begin{cases} a^n u(n)], & \text{if } |z| > a \\ -a^n u(-n-1)], & \text{if } |z| < a \end{cases}$
- $ZT[n a^n u(n)] = \frac{az}{(z-a)^2}, |z| > a \Rightarrow Z^{-1}\left(\frac{az}{(z-a)^2}\right) = n a^n u(n), \text{if } |z| > a$
- $ZT[n(n-1) a^n u(n)] = \frac{a^2 z}{(z-a)^3}, |z| > a \Rightarrow Z^{-1}\left(\frac{a^2 z}{(z-a)^3}\right) = n(n-1) a^n u(n), \text{if } |z| > a$

Example-1: if $X(z) = \frac{N(z)}{(z-p_1)(z-p_2)(z-p_3)}$

$$\Rightarrow \frac{X(z)}{z} = \frac{N(z)}{z(z-p_1)(z-p_2)(z-p_3)}$$

$$\frac{X(z)}{z} = \frac{A}{z} + \frac{B}{z-p_1} + \frac{C}{z-p_2} + \frac{D}{z-p_3}$$

$$X(z) = A + B\left(\frac{z}{z-p_1}\right) + C\left(\frac{z}{z-p_2}\right) + D\left(\frac{z}{z-p_3}\right)$$

$$x(n) = Z^{-1}\left[A + B\left(\frac{z}{z-p_1}\right) + C\left(\frac{z}{z-p_2}\right) + D\left(\frac{z}{z-p_3}\right)\right]$$

$$x(n) = A\, Z^{-1}(1) + B\, Z^{-1}\left(\frac{z}{z-p_1}\right) + C\, Z^{-1}\left(\frac{z}{z-p_2}\right) + D\, Z^{-1}\left(\frac{z}{z-p_3}\right)$$

$$= A\delta(n) + Bp_1^n u(n) + Cp_2^n u(n) + Dp_3^n u(n)$$

Example-2: Determine the right sided or causal sequence x(n) from $X(z)=\frac{z(z+1)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$

$$X(z)=\frac{z(z+1)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

$$\frac{X(z)}{z}=\frac{z+1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

$$=\frac{A}{z-\frac{1}{2}}+\frac{B}{z-\frac{1}{4}}$$

$$A=\frac{\frac{1}{2}+1}{\frac{1}{2}-\frac{1}{4}}=\frac{\frac{3}{2}}{\frac{1}{4}}=6$$

$$B=\frac{\frac{1}{4}+1}{\frac{1}{4}-\frac{1}{2}}=\frac{\frac{5}{2}}{\frac{-1}{4}}=-10$$

$$X(z)=6\left(\frac{z}{z-\frac{1}{2}}\right)-10\left(\frac{z}{z-\frac{1}{4}}\right)$$

For a causal or right sided signal

$|z|>\frac{1}{2}$

and

$|z|>\frac{1}{4}$

$$x(n)=6Z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right)-10Z^{-1}\left(\frac{z}{z-\frac{1}{4}}\right)$$

$$=6\left(\frac{1}{2}\right)^{n}u(n)-10\left(\frac{1}{4}\right)^{n}u(n)$$

Example-3: Determine the left sided or anti-causal sequence x(n) from $X(z)=\frac{z(z+1)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$

$$X(z)=\frac{z(z+1)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

$$=6\left(\frac{z}{z-\frac{1}{2}}\right)-10\left(\frac{z}{z-\frac{1}{4}}\right)$$

For anti-causal or left sided signal

$|z|<\frac{1}{2}$

and

$|z|<\frac{1}{4}$

$$x(n)=6Z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right)-10Z^{-1}\left(\frac{z}{z-\frac{1}{4}}\right)$$

$$=-6\left(\frac{1}{2}\right)^{n}u(-n-1)+10\left(\frac{1}{4}\right)^{n}u(-n-1)$$

Example-4: Determine the both sided sequence x(n) from $X(z)=\frac{z(z+1)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$

$$X(z)=\frac{z(z+1)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

$$=6\left(\frac{z}{z-\frac{1}{2}}\right)-10\left(\frac{z}{z-\frac{1}{4}}\right)$$

$$x(n)=6Z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right)-10Z^{-1}\left(\frac{z}{z-\frac{1}{4}}\right)$$

$$=-6\left(\frac{1}{2}\right)^{n}u(-n-1)-10\left(\frac{1}{4}\right)^{n}u(n)$$

For both sided signal
$|z|<\frac{1}{2}$
and
$|z|>\frac{1}{4}$

Example-5: Determine the right sided or causal sequence x(n) from $X(z)=\frac{z+1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$

$$X(z)=\frac{z+1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

$$\frac{X(z)}{z}=\frac{z+1}{z\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

$$=\frac{A}{z}+\frac{B}{z-\frac{1}{2}}+\frac{C}{z-\frac{1}{4}}$$

$$X(z)=8+12\left(\frac{z}{z-\frac{1}{2}}\right)-20\left(\frac{z}{z-\frac{1}{4}}\right)$$

$$x(n)=8Z^{-1}[1]+12Z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right)-20Z^{-1}\left(\frac{z}{z-\frac{1}{4}}\right)$$

$$=8\delta(n)+12\left(\frac{1}{2}\right)^{n}u(n)-20\left(\frac{1}{4}\right)^{n}u(n)$$

$$A=\frac{0+1}{\left(0-\frac{1}{2}\right)\left(0-\frac{1}{4}\right)}=8$$

$$B=\frac{\frac{1}{2}+1}{\frac{1}{2}\left(\frac{1}{2}-\frac{1}{4}\right)}=\frac{\frac{3}{2}}{\frac{1}{8}}=12$$

$$C=\frac{\frac{1}{4}+1}{\frac{1}{4}\left(\frac{1}{4}-\frac{1}{2}\right)}=\frac{\frac{5}{4}}{\frac{-1}{16}}=-20$$

For a causal or
right sided signal
$|z|>\frac{1}{2}\ \&\ |z|>\frac{1}{4}$

(B)Power Series or Long Division Method:

Partial fraction method is not suitable to evaluate the time domain sequence x(n) when the z-domain X(z) consists of one pole or the factorization of denominator part of X(z) is not possible, to solve such problems, the power series or long division method is used. Process of power series or long division method is given below.

Case 1:

To obtain the causal or right sided sequence, assume x(n) = 0, n < 0.

From the basic definition of z transform

$$ZT[\ x(n)\ ] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \ldots\ldots\ldots\ldots\ldots(1)$$

It is the negative power series expansion of X(z)

From given X(z) = N(z) / D(z), determine the negative power series polynomial by using long division method.

$$X(z) = N(z) / D(z) = a + b\,z^{-1} + c\,z^{-2} + \ldots\ldots\ldots\ldots\ (2)$$

Now compare equations 1 & 2, implies

$$x(n) = \{\underset{\uparrow}{a}, b, c, d, \ldots\ldots\ldots\ldots\ldots\ldots\}$$

It is the sequence representation of required discrete time domain signal

Case 2:

To obtain the anti-causal or left sided sequence, assume x(n) = 0, n > 0.

From the basic definition of z transform

$$ZT[\ x(n)\ ] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{-1} x(n) z^{-n}$$

$$X(z) = x(-1)\ z + x(-2)\ z^{\ 2} + x(-3)\ z^{\ 3} + \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

It is the positive power series expansion of X(z)

From given X(z) = N(z) / D(z), determine the positive power series polynomial by using long division method.

$$X(z) = N(z) / D(z) = a\,z + b\,z^{2} + c\,z^{3} + \ldots\ldots\ldots\ldots\ (2)$$

Now compare equations 1 & 2, implies

$$x(n) = \{\ldots\ldots\ldots\ldots\ldots\ldots, d, c, b, a, \underset{\uparrow}{0}\}$$

Example-1: Determine the right sided or causal sequence x(n) from $X(z) = \dfrac{z+1}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)}$

We know that the negative power series expansion of X(z) is

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \ldots\ldots\ldots\ldots\ldots(1)$$

Given $X(z) = \dfrac{z+1}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)} = \dfrac{z+1}{z^2 - \frac{3}{4}z + \frac{1}{8}}$

Apply long division method and evaluate the negative power series expansion of X(z)

$$z^2 - \frac{3}{4}z + \frac{1}{8} \Big) \; z + 1 \; \Big(z^{-1} + \frac{7}{4}z^{-2} + \frac{19}{16}z^{-3} + \ldots\ldots\ldots.$$

$$\underline{z - \frac{3}{4} + \frac{1}{8}z^{-1}}$$

$$\frac{7}{4} - \frac{1}{8}z^{-1}$$

$$\underline{\frac{7}{4} - \frac{21}{16}z^{-1} - \frac{7}{32}z^{-2}}$$

$$\frac{19}{16}z^{-1} + \frac{7}{32}z^{-2}$$

$$\underline{\frac{19}{16}z^{-1} - \frac{57}{64}z^{-2} + \frac{19}{128}z^{-2}}$$

$$\frac{71}{64}z^{-2} - \frac{19}{128}z^{-2}\ldots\ldots\ldots.$$

$$X(z) = z^{-1} + \frac{7}{4}z^{-2} + \frac{19}{16}z^{-3} + \ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

Compare equations (1) and (2) $\Rightarrow$ $x(0) = 0, x(1) = 1, \; x(2) = \frac{7}{4}, x(3) = \frac{19}{16}, \ldots\ldots\ldots\ldots$

Sequence representation, $x(n) = \left\{ \underset{\uparrow}{0}, \; 1, \; \frac{7}{4}, \; \frac{19}{16}, \ldots\ldots\ldots\ldots \right\}$

Example-2: Determine the left sided or anti-causal sequence x(n) from $X(z) = \dfrac{z+1}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)}$

We know that the positive power series expansion of X(z) is

$$X(z) = x(1)z^{1} + x(2)z^{2} + x(2)z^{3} + \ldots\ldots\ldots\ldots\ldots(1)$$

Given $X(z) = \dfrac{z+1}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)} = \dfrac{z+1}{z^2 - \frac{3}{4}z + \frac{1}{8}}$

Apply long division method and evaluate the positive power series expansion of X(z)

$$\frac{1}{8}-\frac{3}{4}z+z^{2}\Big)\ 1+\ z\ \Big(8+56z+272z^{2}+1184z^{3}+\ldots\ldots\ldots.$$

$$\underline{1\ -\ 6z+\ 8z^{2}}$$

$$7z\ -8z^{2}$$

$$\underline{7z-42z^{2}+56z^{3}}$$

$$34z^{2}\ -56z^{3}$$

$$\underline{34z^{2}-204z^{3}+272z^{4}}$$

$$148z^{3}-272z^{4}\ldots\ldots\ldots.$$

$$X(z)=8+56z+272z^{2}+1184z^{3}+\ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

Compare equations (1) and (2) $\Rightarrow$ $x(0)=8, x(-1)=56,\ x(-2)=272, x(-3)=1184,\ldots\ldots\ldots$

Sequence representation, $x(n)=\{\ldots\ldots\ldots\ldots.1184, 272, 56, \underset{\uparrow}{8}\}$

(C)Residue Method or Contour Integral Method:

If the z-domain X(z) has multiple poles at a single location, then residue or contour integral method is convenient to evaluate discrete time sequence x(n).

$$\text{If } X(z)=\frac{p(z)}{(z-a)^{N}},\ \text{then } x(n)=\frac{1}{(N-1)!}\ \underset{z\to a}{Lt}\left[\frac{d^{N-1}}{dz^{N-1}}\left(p(z)\ z^{n-1}\right)\right]$$

Where,

p(z) : Numerator polynomial of X(z)

z = a : Location of pole

N : Number of poles located at z = a.

Example-1: Determine the causal signal x(n) from the z-domain $X(z)=\dfrac{z}{(z-2)^{3}}$

Given p(z)=z, N = 3 and a =2.

$$x(n)=\frac{1}{(N-1)!}\ \underset{z\to a}{Lt}\left[\frac{d^{N-1}}{dz^{N-1}}\left(p(z)\ z^{n-1}\right)\right]$$

$$=\frac{1}{2!}\ \underset{z\to 2}{Lt}\left[\frac{d^{2}}{dz^{2}}\left(z\ z^{n-1}\right)\right]$$

$$=\frac{1}{2}\ \underset{z\to 2}{Lt}\left[\frac{d^{2}}{dz^{2}}\left(z^{n}\right)\right]$$

$$=\frac{1}{2}\ \underset{z\to 2}{Lt}\left[n\,(n-1)\,z^{n-2}\right]$$

$$=\frac{1}{2}\ n\,(n-1)\,2^{n-2}$$

$$=\frac{n\,(n-1)\,2^{n}}{8}\,u(n)$$

Example-2: Determine the causal signal x(n) from the z-domain $X(z)=\frac{z^2}{(z-2)^3}$

Given p(z)=z^2, N = 3 and a =2.

$$x(n)=\frac{1}{(N-1)!}\ \underset{z\to a}{Lt}\ \left[\frac{d^{N-1}}{dz^{N-1}}\left(p(z)\ z^{n-1}\right)\right]$$
$$=\frac{1}{2!}\ \underset{z\to 2}{Lt}\ \left[\frac{d^2}{dz^2}\left(z^2\ z^{n-1}\right)\right]$$
$$=\frac{1}{2}\ \underset{z\to 2}{Lt}\ \left[\frac{d^2}{dz^2}\left(z^{n+1}\right)\right]$$
$$=\frac{1}{2}\ \underset{z\to 2}{Lt}\ \left[(n+1)\,n\,z^{n-1}\right]$$
$$=\frac{1}{2}\ n\,(n+1)\,2^{n-1}$$
$$=\frac{n\,(n+1)\,2^n}{4}\,u(n)$$

Example-3: Determine the causal signal x(n) from the z-domain $X(z)=\frac{1}{(z-2)^4}$

Given p(z)=1, N = 4 and a =2.

$$x(n)=\frac{1}{(N-1)!}\ \underset{z\to a}{Lt}\ \left[\frac{d^{N-1}}{dz^{N-1}}\left(p(z)\ z^{n-1}\right)\right]$$
$$=\frac{1}{3!}\ \underset{z\to 2}{Lt}\ \left[\frac{d^3}{dz^3}\left(1\ z^{n-1}\right)\right]$$
$$=\frac{1}{6}\ \underset{z\to 2}{Lt}\ \left[\frac{d^3}{dz^3}\left(z^{n-1}\right)\right]$$
$$=\frac{1}{6}\ \underset{z\to 2}{Lt}\ \left[(n-1)\,(n-2)(n-3)\,z^{n-4}\right]$$
$$=\frac{1}{6}\ (n-1)\,(n-2)(n-3)\,2^{n-4}$$
$$=\frac{(n-1)\,(n-2)(n-3)\,2^n}{96}\,u(n)$$

Analysis of Discrete LSI Systems using Z Transform:

Relation between input signal x(n) and output signal y(n) of a discrete time system is called Difference Equation (DE), If the system is Linear Shift Invariant(involves constant coefficients), then it is called Linear Constant Coefficient Difference Equation (LCCDE).

Input Signal, x(n) → Discrete Time System → Output Signal, y(n)

General form of LCCDE is given below

$$y(n)+\sum_{k=1}^{N} a_K\, y(n-k) = \sum_{k=0}^{M} b_K\, x(n-k) \quad \text{or} \quad y(n) = -\sum_{k=1}^{N} a_K\, y(n-k) + \sum_{k=0}^{M} b_K\, x(n-k)$$

(A)Transfer Function:

The ratio between output to input in frequency domain (z-domain) representation is called transfer function or system function. It is represented with H(z) and it can be obtained from LCCDE by applying z transform.

$$ZT[\, y(n)\,] + \sum_{k=1}^{N} ZT[\, a_K\, y(n-k)\,] = \sum_{k=0}^{M} ZT[\, b_K\, x(n-k)\,]$$

$$Y(z) + \sum_{k=1}^{N} a_K\, z^{-k}\, Y(z) = \sum_{k=0}^{M} b_K\, z^{-k}\, X(z)$$

$$Y(z) + Y(z) \sum_{k=1}^{N} a_K\, z^{-k} = X(z) \sum_{k=0}^{M} b_K\, z^{-k}$$

$$Y(z)\, [1 + \sum_{k=1}^{N} a_K\, z^{-k}\,] = X(z) \sum_{k=0}^{M} b_K\, z^{-k}$$

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_K\, z^{-k}}{1 + \sum_{k=1}^{N} a_K\, z^{-k}}$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots\dots\dots\dots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots\dots\dots\dots + a_N z^{-N}}$$

(B)Impulse Response:

Output of discrete LSI system with an input of impulse or unit sample signal is called impulse response or unit sample response.

Input , x(n)=δ(n) → Discrete Time System → Output , y(n)=h(n)

Impulse response is represented with h(n) and it is the inverse z transform of transfer function H(z)

$$h(n) = Z^{-1}[\, H(z)\,]$$

Example: Determine the transfer function and impulse response of discrete LSI system governed by LCCDE $y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2)=x(n-1)+x(n-2)$

Given LCCDE

$$y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2)=x(n-1)+x(n-2)$$

Apply z transform

$$ZT[y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2)]=ZT[x(n-1)+x(n-2)]$$

$$Y(z)-\frac{3}{4}z^{-1}Y(z)+\frac{1}{8}z^{-2}Y(z)=z^{-1}X(z)+z^{-2}X(z)$$

$$Y(z)\left(1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}\right)=X(z)\left(z^{-1}+z^{-2}\right)$$

$$\frac{Y(z)}{X(z)}=\frac{z^{-1}+z^{-2}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$$

$$H(z)=\frac{z+1}{z^2-\frac{3}{4}z+\frac{1}{8}}$$

It is the transfer function of given system.

Apply inverse z transform(Partial fractions method) to get the impulse response, $h(n)=z^{-1}[H(z)]$

$$\text{H(z)}=\frac{z+1}{z^2-\frac{3}{4}z+\frac{1}{8}}$$

$$=\frac{z+1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

$$\frac{H(z)}{z}=\frac{z+1}{z\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

$$=\frac{A}{z}+\frac{B}{z-\frac{1}{2}}+\frac{C}{z-\frac{1}{4}}$$

$$A=\frac{0+1}{\left(0-\frac{1}{2}\right)\left(0-\frac{1}{4}\right)}=8$$

$$B=\frac{\frac{1}{2}+1}{\frac{1}{2}\left(\frac{1}{2}-\frac{1}{4}\right)}=\frac{\frac{3}{2}}{\frac{1}{8}}=12$$

$$C=\frac{\frac{1}{4}+1}{\frac{1}{4}\left(\frac{1}{4}-\frac{1}{2}\right)}=\frac{\frac{5}{4}}{\frac{-1}{16}}=-20$$

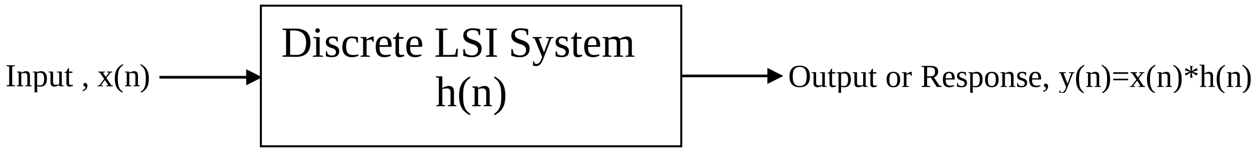
$$H(z)=8+12\left(\frac{z}{z-\frac{1}{2}}\right)-20\left(\frac{z}{z-\frac{1}{4}}\right)$$

$$h(n)=8Z^{-1}[1]+12Z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right)-20Z^{-1}\left(\frac{z}{z-\frac{1}{4}}\right)$$

$$=8\delta(n)+12\left(\frac{1}{2}\right)^n u(n)-20\left(\frac{1}{4}\right)^n u(n)$$

(C)Response of the System:

Response of the discrete LSI system is the convolution of input x(n) and impulse response h(n).



Response of the discrete LSI system can be evaluated by using z transform

$$y(n) = Z^{-1}[\,Y(z)\,], \text{ where } Y(z)=X(z).H(z)$$

Example-1: Determine the response of the discrete LSI system governed by LCCDE $y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2)=x(n-1)+x(n-2)$ with an input of $x(n)=\left(\frac{1}{3}\right)^n u(n)$

Given LCCDE

$$y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2)=x(n-1)+x(n-2)$$

Apply z transform and obtain its transfer function

$$H(z)=\frac{z+1}{z^2-\frac{3}{4}z+\frac{1}{8}}$$

Given input $x(n)=\left(\frac{1}{3}\right)^n u(n) \Rightarrow X(z)=\frac{z}{z-\frac{1}{3}}$

We know that the response y(n) = x(n)*h(n) ⇒Y(z) = X(z)H(z)

$$Y(z)=\frac{z}{z-\frac{1}{3}}\,\frac{z+1}{z^2-\frac{3}{4}z+\frac{1}{8}}$$

$$=\frac{z(z+1)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)\left(z-\frac{1}{4}\right)}$$

$$\frac{Y(z)}{z}=\frac{z+1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)\left(z-\frac{1}{4}\right)}$$

$$=\frac{A}{z-\frac{1}{2}}+\frac{B}{z-\frac{1}{3}}+\frac{C}{z-\frac{1}{4}}$$

$$A=\frac{\frac{1}{2}+1}{\left(\frac{1}{2}-\frac{1}{3}\right)\left(\frac{1}{2}-\frac{1}{4}\right)}=\frac{\frac{3}{2}}{\left(\frac{1}{6}\right)\left(\frac{2}{8}\right)}=36$$

$$B=\frac{\frac{1}{3}+1}{\left(\frac{1}{3}-\frac{1}{2}\right)\left(\frac{1}{3}-\frac{1}{4}\right)}=\frac{\frac{4}{3}}{\left(\frac{-1}{6}\right)\left(\frac{1}{12}\right)}=-96$$

$$C=\frac{\frac{1}{4}+1}{\left(\frac{1}{4}-\frac{1}{2}\right)\left(\frac{1}{4}-\frac{1}{3}\right)}=\frac{\frac{5}{4}}{\left(\frac{-2}{8}\right)\left(\frac{-1}{12}\right)}=60$$

$$Y(z)=36\left(\frac{z}{z-\frac{1}{2}}\right)-96\left(\frac{z}{z-\frac{1}{3}}\right)+60\left(\frac{z}{z-\frac{1}{4}}\right)$$

$$y(n)=36\left(\frac{1}{2}\right)^n u(n)-96\left(\frac{1}{3}\right)^n u(n)+60\left(\frac{1}{4}\right)^n u(n)$$

Example-2: Determine the input required for the discrete LSI system governed by LCCDE $y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2)=x(n-1)-\frac{1}{3}x(n-2)$ to produce the output of

$$y(n)=\left(\frac{1}{2}\right)^n u(n)-\left(\frac{1}{4}\right)^n u(n)$$

Given LCCDE

$$y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2)=x(n-1)-\frac{1}{3}x(n-2)$$

Apply z transform and obtain the transfer function

$$H(z)=\frac{z-\frac{1}{3}}{z^2-\frac{3}{4}z+\frac{1}{8}}=\frac{z-\frac{1}{3}}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

Given output $y(n)=\left(\frac{1}{2}\right)^n u(n)-\left(\frac{1}{4}\right)^n u(n) \Rightarrow Y(z)=\frac{z}{z-\frac{1}{2}}-\frac{z}{z-\frac{1}{4}}=\frac{\frac{1}{4}z}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$

We know that the response y(n) = x(n)*h(n) ⇒Y(z) = X(z)H(z)

$$X(z)=\frac{Y(z)}{H(z)}=\frac{\frac{1}{4}z}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\frac{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}{z-\frac{1}{3}}=\frac{\frac{1}{4}z}{z-\frac{1}{3}}$$

Required input x(n) = Z^{-1}[X(z)]

$$x(n)=Z^{-1}\left(\frac{\frac{1}{4}z}{z-\frac{1}{3}}\right)=\frac{1}{4}Z^{-1}\left(\frac{z}{z-\frac{1}{3}}\right)=\frac{1}{4}\left(\frac{1}{3}\right)^n u(n)$$

Example-3: Determine the response of discrete LSI system having input x(n)={1,2,3,4} and impulse response h(n)={5,6,7,8,9} using z transform.

Given input x(n)={1,2,3,4} ⇒ $X(z)=1+2z^{-1}+3z^{-2}+4z^{-3}$.

Given impulse Response h(n)={5,6,7,8,9}⇒ $H(z)=5+6z^{-1}+7z^{-2}+8z^{-3}+9z^{-4}$.

We know that the system response y(n) = x(n)*h(n)

⇒Y(z) = X(z)H(z)

$= (1+2z^{-1}+3z^{-2}+4z^{-3})(5+6z^{-1}+7z^{-2}+8z^{-3}+9z^{-4})$

$= 5+6z^{-1}+7z^{-2}+8z^{-3}+9z^{-4}+10z^{-1}+12z^{-2}+14z^{-3}+16z^{-4}+18z^{-5}+15z^{-2}+18z^{-3}+21z^{-4}+24z^{-5}+27z^{-6}$
$+20z^{-3}+24z^{-4}+28z^{-5}+32z^{-6}+36z^{-7}$

$=5+16z^{-1}+34z^{-2}+60z^{-3}+70z^{-4}+70z^{-5}+59z^{-6}+36z^{-7}$

Response of the system

y(n) ={5, 16, 34, 60, 70, 70, 59, 36}

Example-4: Determine the input required to produce an output y(n) ={5, 16, 34, 60, 70, 70, 59, 36} of discrete LSI system, given impulse response h(n)={5,6,7,8,9}.

Given output y(n)={5,16,34,60,70,70,59,36}$\Rightarrow Y(z)=5+16z^{-1}+34z^{-2}+60z^{-3}+70z^{-4}+70z^{-5}+59z^{-6}+36z^{-7}$.

Given impulse Response h(n)={5,6,7,8,9}$\Rightarrow H(z)=5+6z^{-1}+7z^{-2}+8z^{-3}+9z^{-4}$.

We know that the response y(n) = x(n)*h(n) $\Rightarrow$Y(z) = X(z)H(z)

$$X(z)=\frac{Y(z)}{H(z)}=\frac{5+16z^{-1}+34z^{-2}+60z^{-3}+70z^{-4}+70z^{-5}+59z^{-6}+36z^{-7}}{5+6z^{-1}+7z^{-2}+8z^{-3}+9z^{-4}}$$

Apply long division method and evaluate the negative power series expansion of X(z)

$$5+6z^{-1}+7z^{-2}+8z^{-3}+9z^{-4})5+16z^{-1}+34z^{-2}+60z^{-3}+70z^{-4}+70z^{-5}+59z^{-6}+36z^{-7}(1+2z^{-1}+3z^{-2}+4z^{-3}$$

$$\underline{5+6z^{-1}+7z^{-2}+8z^{-3}+9z^{-4}+70z^{-5}+59z^{-6}+36z^{-7}}$$

$$10z^{-1}+27z^{-2}+52z^{-3}+61z^{-4}+70z^{-5}+59z^{-6}+36z^{-7}$$

$$\underline{10z^{-1}+12z^{-2}+14z^{-3}+16z^{-4}+18z^{-5}}$$

$$15z^{-2}+38z^{-3}+45z^{-4}+52z^{-5}+59z^{-6}+36z^{-7}$$

$$\underline{15z^{-2}+18z^{-3}+21z^{-4}+24z^{-5}+27z^{-6}}$$

$$20z^{-3}+24z^{-4}+28z^{-5}+32z^{-6}+36z^{-7}$$

$$\underline{20z^{-3}+24z^{-4}+28z^{-5}+32z^{-6}+36z^{-7}}$$

$$0$$

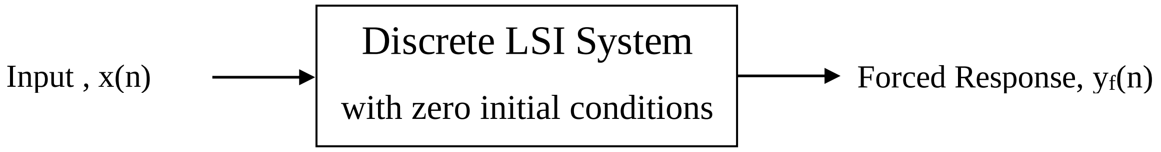
Result of long division method X(z) = $1+2z^{-1}+3z^{-2}+4z^{-3}$

Required input x(n)={1,2,3,4}

(D)Natural and Forced Response:

Response of a discrete LSI system with zero input and for given initial conditions is called zero input response or free response or natural response.

Response of a discrete LSI system by applying input with zero initial conditions is called zero state response or forced response.



Response of a discrete LSI system is the sum of natural response and forced response.

$$\mathbf{y(n) = y_n(n) + y_f(n).}$$

Example-1: Determine (a)Natural response, (b)Forced response and (c)Response of discrete LSI system having LCCDE $y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2)=x(n-1)+x(n-2)$ with input $x(n)=\left(\frac{1}{3}\right)^n u(n)$ and initial conditions y(-1)=y(-2)=1.

(a)Natural response: Take the input x(n)=0 and initial conditions y(-1)=y(-2)=1.

Given LCCDE

$$y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2)=x(n-1)+x(n-2)=0$$

Apply z transform

$$ZT[y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2)]=0$$

$$Y(z)-\frac{3}{4}\left(z^{-1}Y(z)+y(-1)\right)+\frac{1}{8}\left(z^{-2}Y(z)+z^{-1}y(-1)+y(-2)\right)=0$$

$$Y(z)-\frac{3}{4}z^{-1}Y(z)-\frac{3}{4}+\frac{1}{8}z^{-2}Y(z)+\frac{1}{8}z^{-1}+\frac{1}{8}=0$$

$$Y(z)\left(1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}\right)=\frac{5}{8}-\frac{1}{8}z^{-1}$$

$$Y(z)=\frac{\frac{5}{8}-\frac{1}{8}z^{-1}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$$

$$Y(z)=\frac{z\left(\frac{5}{8}z-\frac{1}{8}\right)}{z^2-\frac{3}{4}z+\frac{1}{8}}$$

$$\frac{Y(z)}{z}=\frac{\frac{5}{8}z-\frac{1}{8}}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

$$\frac{Y(z)}{z}=\frac{A}{z-\frac{1}{2}}+\frac{B}{z-\frac{1}{4}}$$

$$A=\frac{\frac{5}{16}-\frac{1}{8}}{\frac{1}{2}-\frac{1}{4}}=\frac{\frac{3}{16}}{\frac{1}{4}}=\frac{3}{4}$$

$$B=\frac{\frac{5}{32}-\frac{1}{8}}{\frac{1}{4}-\frac{1}{2}}=\frac{\frac{1}{32}}{\frac{-1}{4}}=\frac{-1}{8}$$

$$Y(z)=\frac{3}{4}\left(\frac{z}{z-\frac{1}{2}}\right)-\frac{1}{8}\left(\frac{z}{z-\frac{1}{4}}\right)$$

$$y(n)=\frac{3}{4}\left(\frac{1}{2}\right)^n-\frac{1}{8}\left(\frac{1}{4}\right)^n, n\geq -2$$

Natural Response $y_n(n)=y(n)=\frac{3}{4}\left(\frac{1}{2}\right)^n-\frac{1}{8}\left(\frac{1}{4}\right)^n, n\geq -2$

(b)Forced response: Apply the input $x(n) = \left(\frac{1}{3}\right)^n u(n)$ without initial conditions.

Given LCCDE

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n-1) + x(n-2)$$

Apply z transform

$$ZT[y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2)] = ZT[x(n-1) + x(n-2)]$$

$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = \left(1 + z^{-1}\right)X(z)$$

$$Y(z)\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right) = \left(1 + z^{-1}\right)X(z)$$

$$Y(z) = \frac{\left(1 + z^{-1}\right)X(z)}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \left(\frac{z+1}{z^2 - \frac{3}{4}z + \frac{1}{8}}\right)\left(\frac{z}{z - \frac{1}{3}}\right)$$

$$\frac{Y(z)}{z} = \frac{z+1}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)\left(z - \frac{1}{4}\right)}$$

$$\frac{Y(z)}{z} = \frac{A}{z - \frac{1}{2}} + \frac{B}{z - \frac{1}{3}} + \frac{C}{z - \frac{1}{4}}$$

$$A = \frac{\frac{1}{2} + 1}{\left(\frac{1}{2} - \frac{1}{3}\right)\left(\frac{1}{2} - \frac{1}{4}\right)} = \frac{\frac{3}{2}}{\left(\frac{1}{6}\right)\left(\frac{2}{8}\right)} = 36$$

$$B = \frac{\frac{1}{3} + 1}{\left(\frac{1}{3} - \frac{1}{2}\right)\left(\frac{1}{3} - \frac{1}{4}\right)} = \frac{\frac{4}{3}}{\left(\frac{-1}{6}\right)\left(\frac{1}{12}\right)} = -96$$

$$C = \frac{\frac{1}{4} + 1}{\left(\frac{1}{4} - \frac{1}{2}\right)\left(\frac{1}{4} - \frac{1}{3}\right)} = \frac{\frac{5}{4}}{\left(\frac{-2}{8}\right)\left(\frac{-1}{12}\right)} = 60$$

$$Y(z) = 36\left(\frac{z}{z - \frac{1}{2}}\right) - 96\left(\frac{z}{z - \frac{1}{3}}\right) + 60\left(\frac{z}{z - \frac{1}{4}}\right)$$

$$y(n) = 36\left(\frac{1}{2}\right)^n - 96\left(\frac{1}{3}\right)^n + 60\left(\frac{1}{4}\right)^n, n \ge 0$$

Forced Response, $y_f(n) = y(n) = 36\left(\frac{1}{2}\right)^n - 96\left(\frac{1}{3}\right)^n + 60\left(\frac{1}{4}\right)^n, n \ge 0$

(c)Response of a discrete LSI system is the sum of natural response and forced response.

$$y(n) = y_n(n) + y_f(n)$$

$$y(n) = \frac{3}{4}\left(\frac{1}{2}\right)^n - \frac{1}{8}\left(\frac{1}{4}\right)^n + 36\left(\frac{1}{2}\right)^n - 96\left(\frac{1}{3}\right)^n + 60\left(\frac{1}{4}\right)^n, n \ge -2$$

$$y(n) = \frac{147}{4}\left(\frac{1}{2}\right)^n - 96\left(\frac{1}{3}\right)^n - \frac{479}{8}\left(\frac{1}{4}\right)^n, n \ge -2$$

(E)Causality and Stability of the System:

- A discrete LSI system having the rational system function H(z) is said be causal only when the degree of numerator can't be greater than the degree of denominator and the ROC should be outside the circle of outer most pole.

- A discrete LSI system having the rational system function H(z) is said to be stable only when the ROC includes the unit circle.

- A discrete LSI causal system having the rational system function H(z) is said to be stable only when the poles of H(z) should be inside the unit circle.

Examples: Analyze the following discrete LSI systems for causality and stability.

$$(1)\, H(z) = \frac{z(z^4+1)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}$$

- Given discrete LSI system is non causal because the degree of numerator is more when compared to the degree of denominator.
- Given discrete LSI system is stable if the ROC is $|z| > \frac{1}{2}$.

$$(2)\, H(z) = \frac{z(z+1)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}$$

- Given discrete LSI system is causal if the ROC is $|z| > \frac{1}{2}$.
- Given discrete LSI system is stable if the ROC is $|z| > \frac{1}{2}$.

$$(3)\, H(z) = \frac{z(z+1)}{(z-2)(z-3)(z-4)(z-5)}$$

- Given discrete LSI system is causal if the ROC is $|z| > 5$.
- Given discrete LSI system is stable if the ROC is $|z| < 2$.

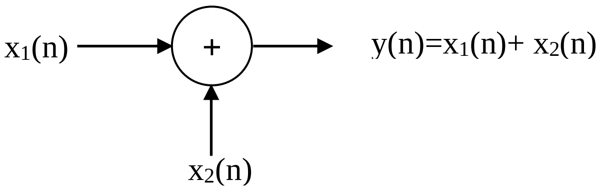
$$(4)\, H(z) = \frac{z(z+1)}{\left(z-\frac{1}{2}\right)(z-3)(z-4)(z-5)}$$

- Given discrete LSI system is causal if the ROC is $|z| > 5$.
- Given discrete LSI system is stable if the ROC is $\frac{1}{2} < |z| < 3$.

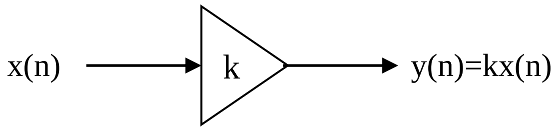
Realization of Discrete Time Systems:

Physical representation or implementation or design of a discrete LSI system by using discrete components, like adders, constant multipliers and delays (memories) is called realization.

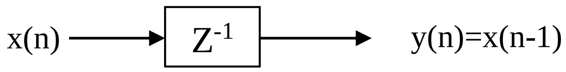
Adder is used to add two or more signals, for example $y(n) = x_1(n) + x_2(n)$



Constant multiplier is used to get the product of a constant and a signal, for example $y(n)=kx(n)$



Delay or Memory unit is used to get one unit delay of a signal, for example $y(n) = x(n - 1)$



Various structures of realization methods or techniques , which are used in the design of Infinite Impulse Response (IIR) & Finite Impulse Response (FIR) systems are given below.

Realization methods in IIR System:

- Direct Form–I Realization
- Direct Form–II or Canonic Form Realization
- Cascade Form Realization
- Parallel Form Realization

Realization methods in FIR System:

- Direct Form Realization
- Cascade Form Realization
- Linear Phase Realization

Choice of realization structure depends on computational complexity and memory requirements.

- Computational complexity refers to the number arithmetic operations (additions & multiplications) required to compute the response.
- Memory requirements refer to the number of memory locations required to store the system parameters (present input, past inputs, intermediate computed values and outputs).

(A)Direct Form– I (DF-I) Realization of IIR System:

From the general form of system function or transfer function of IIR system

$$H(z)=\frac{N(z)}{D(z)}=\frac{Y(z)}{X(z)}=\frac{b_0+b_1z^{-1}+b_2z^{-2}+.................+b_Mz^{-M}}{1+a_1z^{-1}+a_2z^{-2}+.................+a_Nz^{-N}}$$

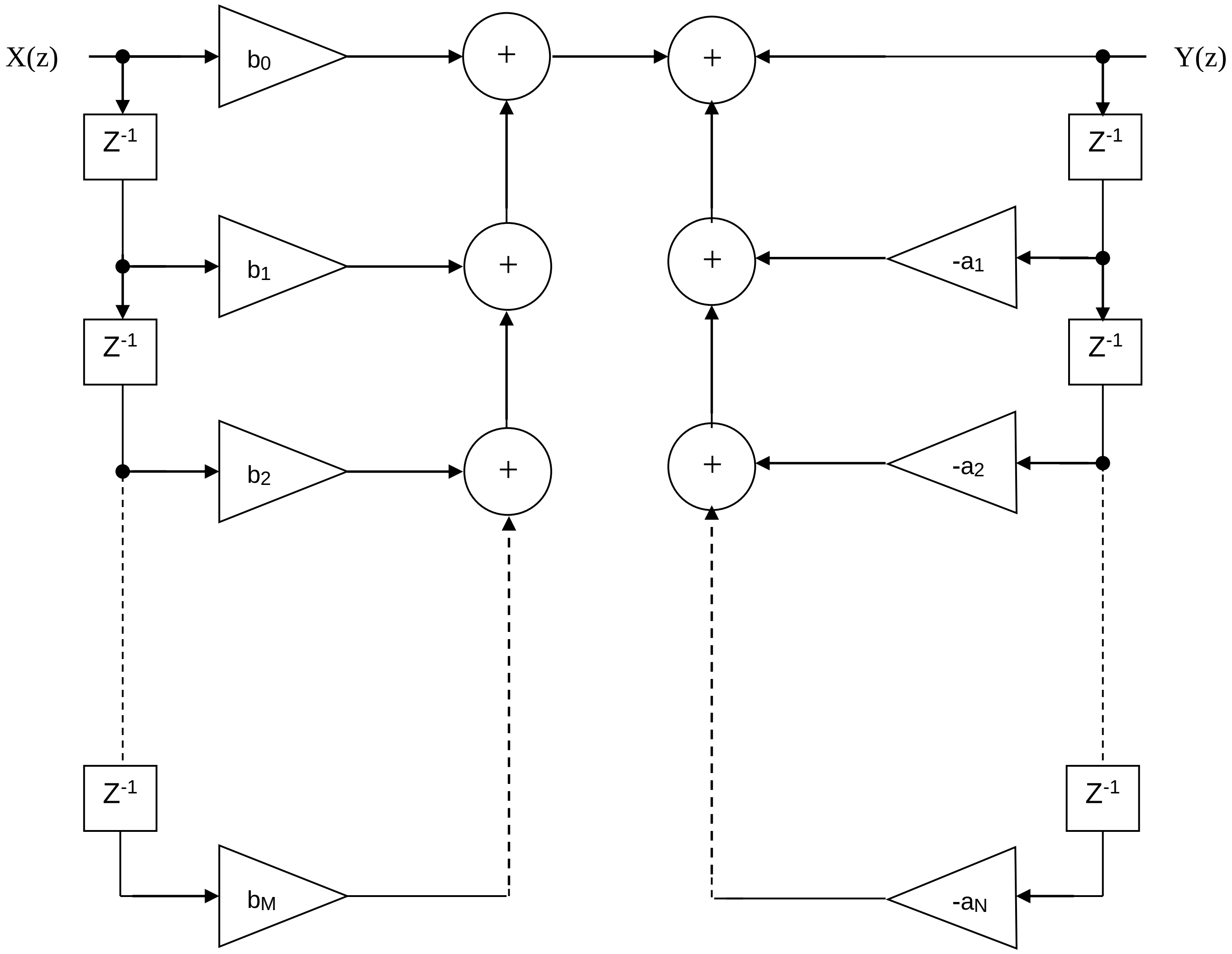
$$\Rightarrow (1+a_1z^{-1}+a_2z^{-2}+.................+a_Nz^{-N})Y(z)=(b_0+b_1z^{-1}+b_2z^{-2}+.................+b_Mz^{-M})X(z)$$

$$\Rightarrow Y(z)+a_1z^{-1}Y(z)+a_2z^{-2}Y(z)+.................+a_Nz^{-N}Y(z)=$$
$$b_0X(z)+b_1z^{-1}X(z)+b_2z^{-2}X(z)+.................+b_Mz^{-M}X(z)$$

$$\Rightarrow Y(z)=-a_1z^{-1}Y(z)-a_2z^{-2}Y(z)-.................-a_Nz^{-N}Y(z)+I(z)-----------------(1)$$

$$\text{where,}\quad I(z)=b_0X(z)+b_1z^{-1}X(z)+b_2z^{-2}X(z)+.................+b_Mz^{-M}X(z)-----------(2)$$

Draw the realization structure by using equations (1) and (2)



Direct Form-I structure is the basic form of realization and it involves

- 'M+N' number of adders.
- 'M+N+1' number of constant multipliers.
- 'M+N' number of delays.

(B)Direct Form– II (DF-II) Realization of IIR System:

From the general form of system function or transfer function of IIR system

$$H(z) = \frac{N(z)}{D(z)} = \frac{Y(z)}{X(z)} = \left(\frac{Y(z)}{I(z)}\right)\left(\frac{I(z)}{X(z)}\right) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots\dots\dots\dots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots\dots\dots\dots + a_N z^{-N}}$$

$$\Rightarrow \frac{Y(z)}{I(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots\dots\dots\dots + b_M z^{-M}$$

$$\Rightarrow Y(z) = (b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots\dots\dots\dots + b_M z^{-M}) I(z)$$

$$\Rightarrow Y(z) = b_0\, I(z) + b_1 z^{-1}\, I(z) + b_2 z^{-2}\, I(z) + \dots\dots\dots\dots + b_M z^{-M}\, I(z) \text{ - - - - - - - - - (1) and}$$

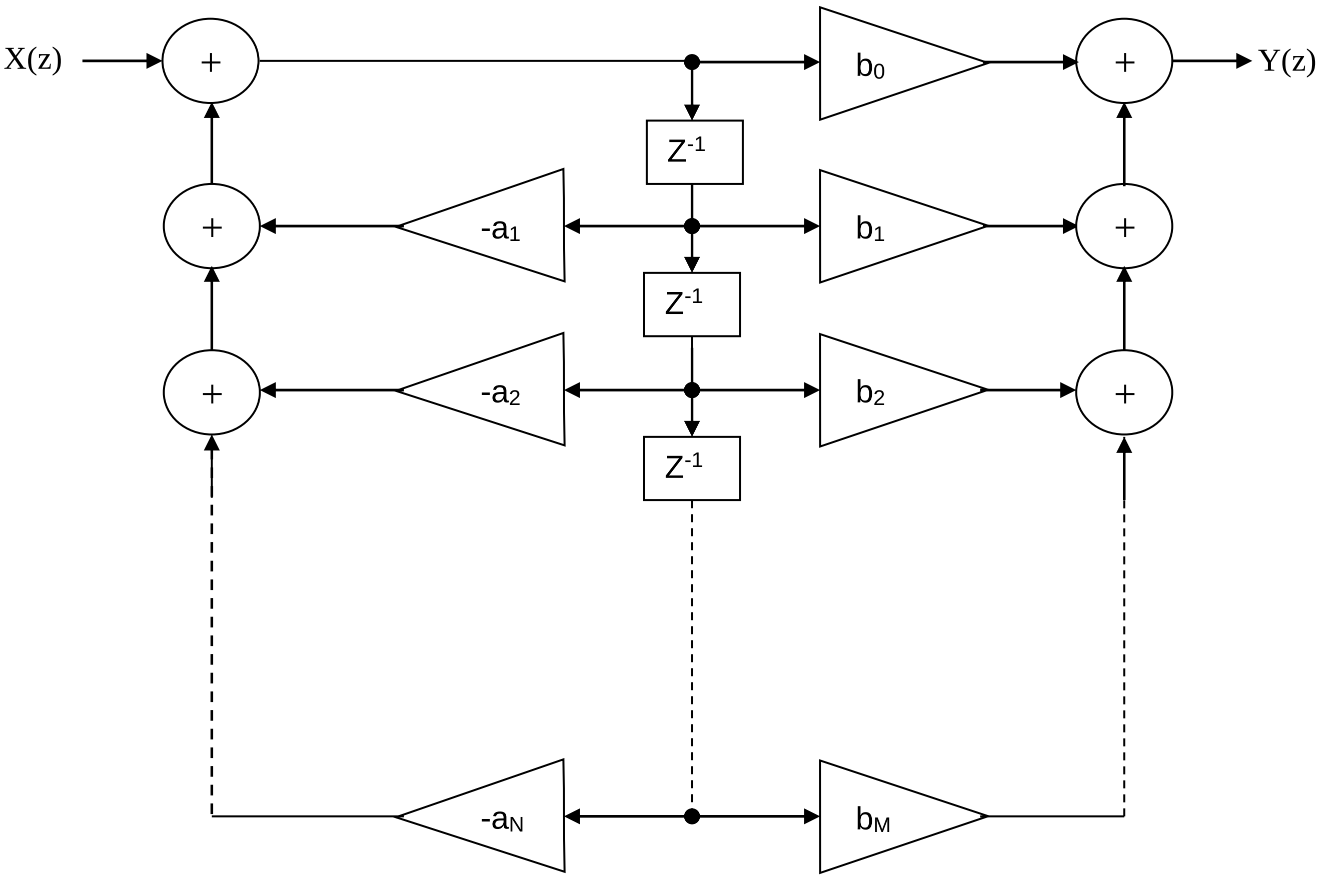
$$\Rightarrow \frac{I(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots\dots\dots\dots + a_N z^{-N}}$$

$$\Rightarrow I(z)\,(1 + a_1 z^{-1} + a_2 z^{-2} + \dots\dots\dots\dots + a_N z^{-N}) = X(z)$$

$$\Rightarrow I(z) + a_1 z^{-1} I(z) + a_2 z^{-2} I(z) + \dots\dots\dots\dots + a_N z^{-N} I(z) = X(z)$$

$$\Rightarrow I(z) = X(z) - a_1 z^{-1} I(z) - a_2 z^{-2} I(z) - \dots\dots\dots\dots - a_N z^{-N} I(z) \text{ - - - - - - - - - - - - (2)}$$

Draw the realization structure by using equations (1) and (2)



Direct Form-II structure is also known as Canonic Form realization. It is the modified form of Direct Form-I realization and it involves

- ‘M+N’ number of adders.
- ‘M+N+1’ number of constant multipliers.
- ‘M’ number of delays if M>N and ‘N’ number of delays if N>M.

It is most widely used realization, because it involves less number of delays.

(C)Cascade Form of Realization of IIR System:

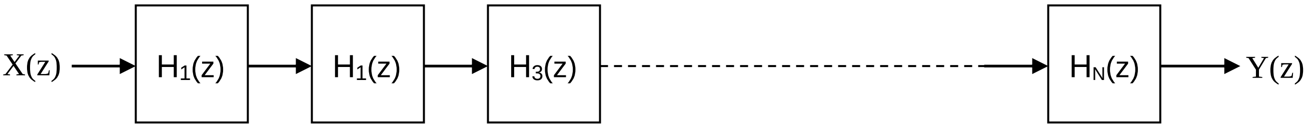
If the transfer function of discrete LSI System is the product of $H_1(z)$, $H_2(z)$, $H_3(z)$. …… $H_N(z)$, then the cascede form of realization is used.

Given $H(z) = H_1(z) . H_2(z) . H_3(z). ………… H_N(z)$

$\Rightarrow Y(z) / X(z) = H_1(z) . H_2(z) . H_3(z). ………… H_N(z)$

$\Rightarrow Y(z) = H_1(z) . H_2(z) . H_3(z). ………… H_N(z). X(z)$

Draw the realization structure by using above equation.



- It is the cascading of $H_1(z)$, $H_2(z)$, $H_3(z)$. …… $H_N(z)$.
- Realize $H_1(z)$, $H_2(z)$, $H_3(z)$. …… $H_N(z)$ by using canonic form of realization.
- $H_1(z)$, $H_2(z)$, $H_3(z)$. …… $H_N(z)$ are first or second order transfer functions

(D)Parallel Form of Realization of IIR System:

If the transfer function of discrete LSI System is the sum of $H_1(z)$, $H_2(z)$, $H_3(z)$. …… $H_N(z)$, then the parallel form of realization is used.

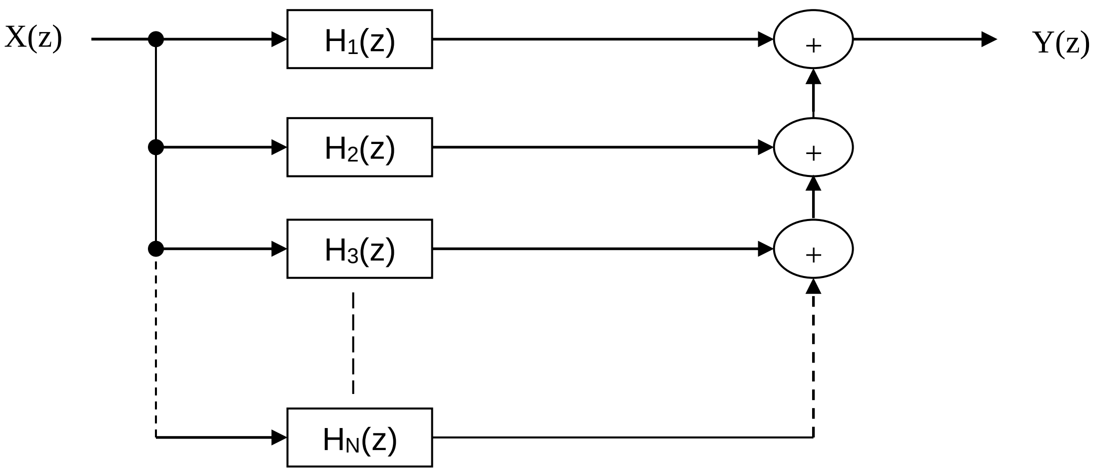
Given $H(z) = H_1(z) + H_2(z) + H_3(z)+ ………..+ H_N(z)$

$\Rightarrow Y(z) / X(z) = H_1(z) + H_2(z) + H_3(z)+ ………..+ H_N(z)$

$\Rightarrow Y(z) = [H_1(z) + H_2(z) + H_3(z)+ ………..+ H_N(z)] X(z)$

$\Rightarrow Y(z) = H_1(z) X(z) + H_2(z) X(z) + H_3(z) X(z) + ………..+ H_N(z) X(z)$

Draw the realization structure by using above equation.



- It is the parallel connection of $H_1(z)$, $H_2(z)$, $H_3(z)$. …… $H_N(z)$.
- Realize $H_1(z)$, $H_2(z)$, $H_3(z)$. …… $H_N(z)$ by using canonic form of realization.
- $H_1(z)$, $H_2(z)$, $H_3(z)$. …… $H_N(z)$ are first or second order transfer functions

(E)Direct Form Realization of FIR System:

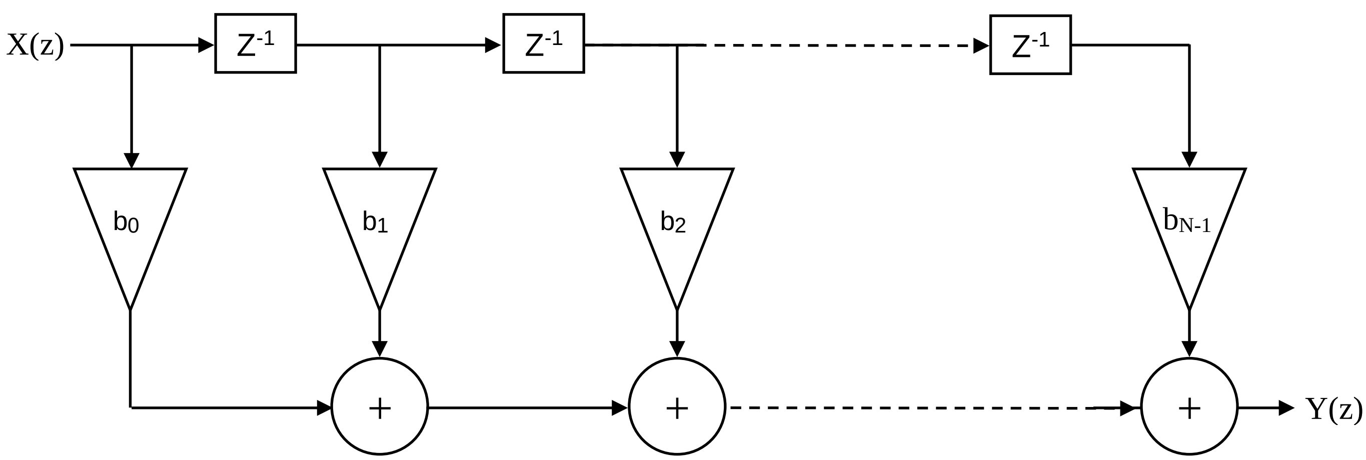
From the general form of system function or transfer function of FIR system

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots\ldots\ldots\ldots + b_{N-1} z^{-(N-1)}$$

$$\Rightarrow Y(z) = (b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots\ldots\ldots\ldots + b_{N-1} z^{-(N-1)}) X(z)$$

$$\Rightarrow Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \ldots\ldots\ldots\ldots + b_{N-1} z^{-(N-1)} X(z)$$

Draw the realization structure by using above equation.



Direct Form realization of FIR system involves

- 'N-1' number of adders.
- 'N' number of constant multipliers.
- 'N-1' number of delays.

(F)Cascade Form Realization of FIR System:

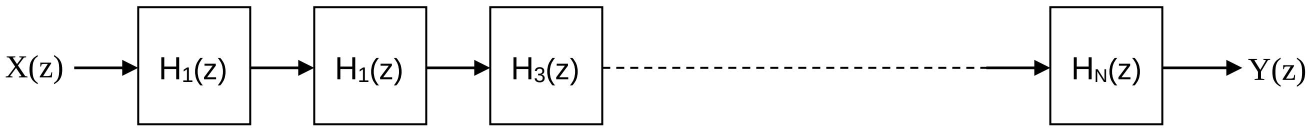
If the transfer function of discrete LSI System is the product of $H_1(z)$, $H_2(z)$, $H_3(z)$. …… $H_N(z)$, then the cascede form of realization is used.

Given $H(z) = H_1(z) . H_2(z) . H_3(z). \ldots\ldots\ldots\ldots H_N(z)$

$\Rightarrow Y(z) / X(z) = H_1(z) . H_2(z) . H_3(z). \ldots\ldots\ldots\ldots H_N(z)$

$\Rightarrow Y(z) = H_1(z) . H_2(z) . H_3(z). \ldots\ldots\ldots\ldots H_N(z). X(z)$

Draw the realization structure by using above equation.



(G)Linear Phase Realization of FIR System:

From the general form of system function or transfer function of FIR system

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots\ldots\ldots\ldots\ldots + b_{N\text{-}1} z^{-(N-1)}$$

$$\Rightarrow Y(z) = (b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots\ldots\ldots\ldots\ldots + b_{N\text{-}1} z^{-(N-1)})X(z)$$

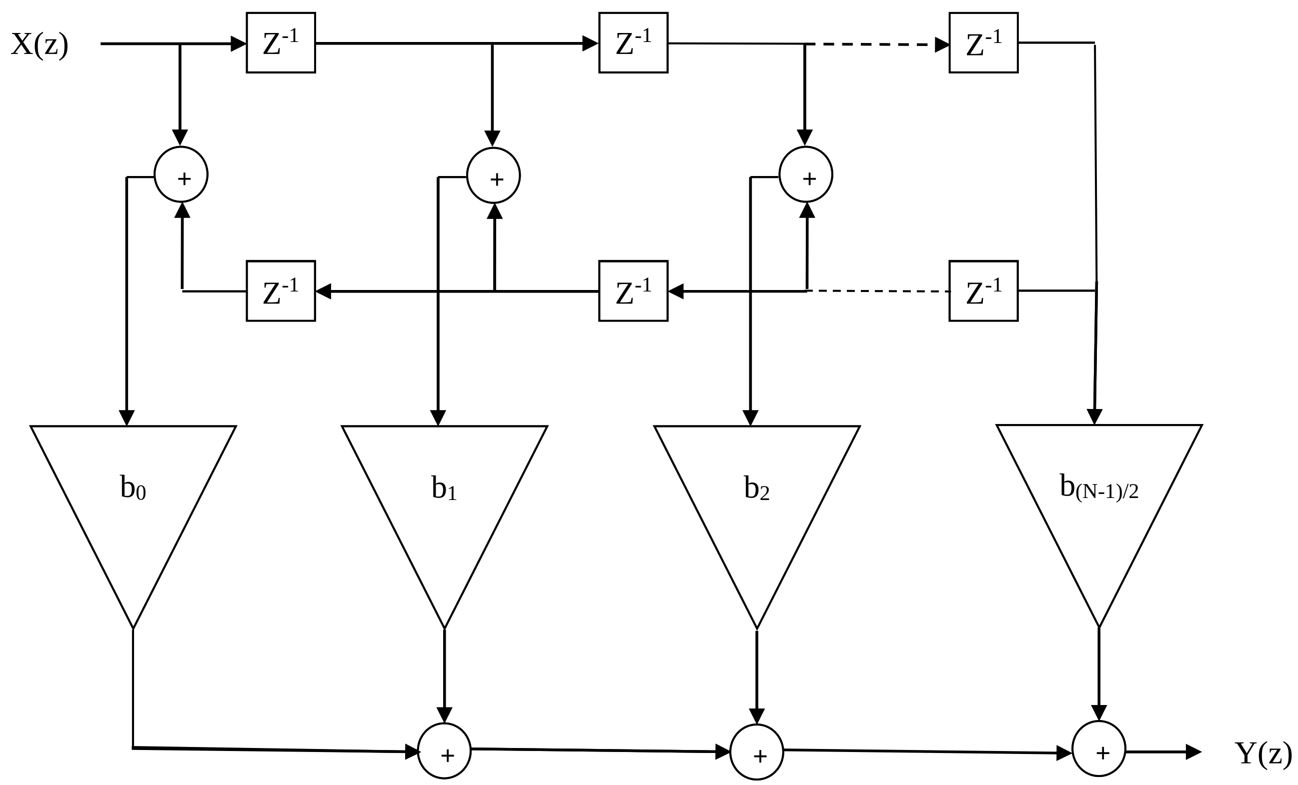
$$\Rightarrow Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \ldots\ldots\ldots\ldots\ldots + b_{N\text{-}1} z^{-(N-1)} X(z)$$

Linear phase realization is possible only when $b_0 = b_{N-1}$, $b_1 = b_{N-2}$, $b_2 = b_{N-3}$, …………,

$$\Rightarrow Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \ldots\ldots\ldots\ldots + b_2 z^{-(N-3)} X(z) + b_1 z^{-(N-2)} X(z) + b_0 z^{-(N-1)} X(z)$$

$$\Rightarrow Y(z) = b_0 (1 + z^{-(N-1)}) X(z) + b_1 (z^{-1} + z^{-(N-2)}) X(z) + b_2 (z^{-2} + z^{-(N-3)}) X(z) + \ldots\ldots\ldots$$

Draw the realization structure by using above equation.



Linear Phase Realization of FIR system involves

- ‘N-1’ number of adders.
- ‘N’ number of constant multipliers.
- ‘(N-1)/2’ number of delays.

Descriptive Questions:

1. Evaluate the Z-Transform and indicate the ROC for the following sequences (i)$x(n)=a^n u(n)$ (ii)$y(n)=-a^n u(-n-1)$ (iii)$z(n)=a^{|n|}$
2. Evaluate the Z-Transform and indicate the ROC for the following sequences

   (i) $x(n)=3\left(\frac{1}{2}\right)^n u(n)-2\left(\frac{1}{3}\right)^n u(n)$. (ii) $x(n)=3\left(\frac{1}{2}\right)^n u(-n-1)-2\left(\frac{1}{3}\right)^n u(-n-1)$

   (iii) $x(n)=3\left(\frac{1}{2}\right)^n u(-n-1)+2\left(\frac{1}{3}\right)^n u(n)$ (iv) $x(n)=3\left(\frac{1}{2}\right)^n u(n)+2\left(\frac{1}{3}\right)^n u(-n-1)$
3. Evaluate the Z-Transform and indicate the ROC for the following sequences

   (i) $x(n)=a^n Sin(n\theta)u(n)$ (ii) $y(n)=a^n Cos(n\theta)u(n)$
4. Apply the properties to Evaluate the Z-Transform and associate ROC for the following sequences (i)$x(n)=n^2a^n$ u(n) (ii) $x(n)=a^n$ u(-n) (iii) $x(n)=a^{n+10}$ u(n-10)
5. Apply partial fractions method and compute all possible cases of sequences from given X(z)

   (i) $X(z)=\frac{z}{(z-1/2)(z-1/4)}$ (ii) $X(z)=\frac{z+1}{(z-1/2)(z-1/4)}$ (iii) $X(z)=\frac{z}{(z-1/2)^2(z-1/4)}$
6. Apply power series method and Evaluate both the causal and Non-causal sequences from

   $$X(z)=\frac{z^2+z+1}{2z^3+3z^2+z+4}$$
7. Apply Residue method and Obtain the causal sequence x(n) from $X(z)=\frac{z(z-2)}{(z-1/4)^3}$
8. Compute the initial value of a causal sequence x(n) from the z-domain

   $$X(z)=\frac{z(z+1)(z+2)}{(2z-1/2)(4z-1/4)(8z-1/8)}$$
9. Calculate the final value of a causal sequence x(n) from the z-domain

   $$X(z)=\frac{z(z+1)(z+2)}{(z-1)(z-1/2)(z-1/4)(z-1/8)}$$
10. Determine the z-transform of a convoluted sequence

    (a) $x(n)=u(n)*nu(n)*n^2u(n)$ (b) $x(n)=n(1/2)^{n+1}u(n-1)$
11. Find the Inverse z-transform $X(z)=\log(1+az^{-1})$
12. Realize the discrete system having LCCDE y(n)-1/2y(n-1)-1/3y(n-2)=2x(n)+3x(n-1)+4x(n-2) by using (i)DF-I realization (ii)DF-II realization

13. Find a causal sequence from the z-domain

(i) $X(z)=\frac{1}{z-1}$ (ii) $X(z)=\frac{1}{z^2(z-1)}$ (iii) $X(z)=\frac{1}{z^{99}(z-99)}$

14. Apply z-Transform to solve the difference equation

y(n)-ay(n-1)=x(n) and obtain the impulse response, hence obtain the system response for an input x(n)=b^nu(n), where 0<a<1 and 0<b<1.

15. Apply z-Transform to obtain (a)Unit sample response (b)Unit step response and (c)Response of a causal system for an input x(n)=$(1/2)^n$u(n).

Given LCCDE y(n)-0.25y(n-1)=x(n).

16. Apply z-Transform to find the response of the system for an input x(n)=$(1/4)^n$u(n) having LCCDE y(n)-5/6 y(n-1)+1/6 y(n-2)=x(n)+x(n-1). Given initial conditions y(-1)=y(-2)=1.

17. Apply z-Transform for the system having LCCDE y(n)-5/6 y(n-1)+1/6 y(n-2)=x(n)+x(n-1), find (a)Natural response for given initial conditions y(-1)=y(-2)=1 (b)Forced response for given input x(n)=$(1/2)^n$u(n) and (c)Response of the system for given input x(n)=$(1/2)^n$u(n) and initial conditions y(-1)=y(-2)=1.

18. Apply z-Transform to find the natural response of the system having LCCDE y(n)-2 y(n-1)+4 y(n-2)=x(n)+x(n-1) for given initial conditions y(-1)=y(-2)=1.

19. Analyze the following systems for causality and stability

(a) $X(z)=\frac{z(z+1)(z+2)}{(z-1)(z-1/2)(z-1/4)(z-1/8)}$ (b) $X(z)=\frac{z(z+1)(z+2)}{(2z-1/2)(4z-1/4)(8z-1/8)}$

20. Realize the discrete system having LCCDE y(n)-5/6 y(n-1)+1/6 y(n-2)=x(n)+x(n-1).

(a)Canonic Form (b)Cascade Form (c)Parallel Form

Quiz Questions:

1.	What is the ROC of ZT[u(n)] (A)|z|>0 (B)|z|>1 (C)0<|z|<1 (D)|z)<0	B
2.	Find the z-domain of x(n)={1,0,1} (A)$1+z^{-1}$ (B) $z+1+z^{-1}$ (C) $z+z^{-1}$ (D) 1+z	A
3.	Determine the Z-transform of x(n)=δ(n-9)	z^{-9}
4.	What is the ROC of z transform of " $(\frac{1}{2})^n u(n) + (\frac{1}{3})^n u(n) + u(n)$ "	|z|>1
5.	If ZT of $a^n u(n) * b^n u(n)$ is X(z)/(z-a)(z-b), then X(z) is (A)1 (B) z (C) z^2 (D) 1+z	C
6.	If ZTof u(-n) is k/X(z), then k-X(z) is	z
7.	Find a causal sequence from the z-domain $X(z) = \frac{1}{z-1}$	u(n-1)
8.	If $X(z) = \frac{9}{z^9(z-9)}$ and $x(n)=a^{n-b}u(n-c)$, then a,b,c= (A)9,9,9 (B) 9,10,9 (C) 10,9,9 (D) 9,9,10	D
9.	Find the initial value of a sequence x(n) from $X(z) = \frac{z(z+1)}{(3z-1)(2z-1)}$	1/6
10.	Find the final value of a sequence x(n) from $X(z) = \frac{z(z+1)}{(z-1)(2z-1)}$	2